A simple explicit model approximating the relationship between speed and density of vehicular traffic on urban roads

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Abstract: With the increase in simulation of urban environments for the purpose of planning, modelling vehicular traffic has become important. Since empirical evidence on traffic flow is relatively sparse, models are being increasingly used for planning urban roads and environments. In this paper, a simple, explicit model is proposed to approximate the speed versus density of vehicular traffic flow. The model, which uses two parameters derived from simple measurements of real-time traffic data, allows for a prediction of the approximate relationship for congested as well as uncongested vehicular traffic flow. The proposed model is especially useful in conditions where available data is sparse and can be invaluable for the modelling and simulation of urban environments.

Keywords: vehicular traffic; speed-density modelling; explicit model.


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1 Introduction

Modelling vehicular traffic for the purpose of planning roads and/or other urban infrastructure has become pervasive in recent times (Report on Indian Urban Infrastructure and Services, 2011; Ban et al., 2010; Haggerty et al., 2008; Lin et al., 2008). As a result, it is very important that the models are used effectively and accurately represent the characteristics of the roads being modelled. However, empirical data on traffic flow is relatively sparse. Collection of real-world data is time-consuming, expensive, and cumbersome. It is thus important that the models being used to estimate traffic flow are not data intensive, yet sufficiently represent the phenomena of interest.

In this paper, we propose a simple, explicit model that relates the vehicular speed to the vehicular density on a given road segment. Our proposed model is of a general form and the parameters can be estimated with a very sparse set of data. We demonstrate how this model can be used with an existing real-world dataset and how it is reasonable in its approximation.

Our paper is structured as follows: Section 2 describes the previous work in this area. In Section 3, we describe our model, and Section 4 compares our model to the existing models. Section 5 describes how the parameter values are to be estimated from sample data. In Section 6, we use some real-world data to explain our model and how it can be used. We conclude with some remarks and caveats about our proposed model.

2 Previous work

A generic traffic flow model can be represented as

\[ \partial_t \rho + \partial_x f(\rho) = 0, \quad x \in \mathbb{R}, \ t > 0 \]
\[ \rho(x, 0) = \rho_0(x), \quad x \in \mathbb{R} \]

where \( \rho \) is the traffic density and \( f(\rho) \) can be viewed as the outward flux of the scalar density function \( \rho \). The ‘flow’ parameter of traffic model, which can be defined as the number of vehicles passed at fixed point in unit time, can be represented as the outward flux \( f(\rho) \). Due to Edie (1965), for stationary traffic, the flow parameter, \( f(\rho) \) equals space-mean speed times density, i.e., \( f(\rho) = \rho v \), where \( v \) is the space mean speed. Traffic researchers have long been interested in functionally specifying and estimating these relations (Greenshield, 1935; Drake et al., 1967). Greenshield’s (1935) data suggested a linear speed-density relation, leading him to propose a parabolic function as an approximation to the flow-density relation. Other functional forms, based on notions like fluid dynamics and car-following decisions, give rise to a variety of forms. Considering the free-flow speed (which can be considered as the maximum speed) as \( v_f \), the maximum density \( \rho_{\text{max}} \), Lighthill-Whitham-Richards linear model (Lighthill and Whitham, 1955) can be represented as

\[ v(\rho) = v_f \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \text{ for } 0 \leq \rho \leq \rho_{\text{max}} \]  

Greenberg (1959) proposed a logarithmic form for speed versus density,
We can consider that at any instant the traffic density ($\rho_0$) produces the traffic speed as $v_0$. Then the Greenshield (1935) linear model can be expressed as

$$v(\rho) = v_0 \left( 2 - \frac{\rho}{\rho_0} \right) \text{ for } 0 < \rho \leq \rho_{\text{max}}$$

(3)

Underwood (1961) used an exponential form,

$$v(\rho) = v_0 \exp \left(1 - \frac{\rho}{\rho_0} \right) \text{ for } 0 < \rho \leq \rho_{\text{max}}$$

(4)

Greenberg (1959) used the logarithmic form also as

$$v(\rho) = v_0 \left(1 - \ln \frac{\rho}{\rho_0} \right) \text{ for } 0 < \rho \leq \rho_{\text{max}}$$

(5)

Drake et al. (1967) proposed the model resembling Underwood (1961) as

$$v(\rho) = v_0 \exp \left(1 - \frac{\rho^2}{2 (1 - \frac{\rho}{\rho_0})} \right) \text{ for } 0 < \rho \leq \rho_{\text{max}}$$

(6)

The ‘stimulus-response’ car-following models of Gazis et al. (1961) have a general form in which the acceleration of a following vehicle responds to the separation and difference in speed from the vehicle in front, but this model is complex due to the presence of space derivatives. Papacostas (1987) proposed a model expressible as an extension of Duncan’s (1976, 1979) model, Newell’s model (Zhang and Kim, 2005), and a somewhat rearranged version of Underwood’s (1961) model with the additional equation on hypothetical speed at zero density. Random driver behaviours are analyzed in Krauss (1997) to investigate traffic variability, and Kockelman (1998) uses the interacting information on travellers, weather, and vehicle type with density for a least-squares polynomial model of flow. Laval and Leclercq (2008) describe their model through a merge where traffic reached unsustainable flow and density values, which both gradually ‘relaxed’ to lower values, in an almost orthogonal direction to the steady-state flow-density relationship. Some researchers also used microscopic modelling of the relaxation phenomenon using a macroscopic lane-changing model (Laval and Leclercq, 2008). Addison and Heydecker (2008) use different traffic speed limits in their model which found that jam density is not a property of moving traffic.

In this paper, we propose a new model to approximate the speed versus density relationship of traffic flow. The functional relationship between the speed and traffic density is simple and explicit and shows the average trend of these two basic traffic variables.

### 3 Approximate explicit model for speed-density curve

The proposed model for the speed versus density of the traffic flow can be represented as
\[
\frac{v}{v_{\text{max}}} = 1 - (1 - m) \frac{\rho}{\rho_{\text{max}}} - m \left( \frac{\rho}{\rho_{\text{max}}} \right)^n
\]  
(7)

for \(0 \leq \rho \leq \rho_{\text{max}}\), where \(m\) and \(n\) are two vehicle related parameters depending on the statistical behaviour of the traffic. Here when \(\rho \to \rho_{\text{max}}\), \(v \to 0\) and for \(\rho \to 0\), \(v \to v_{\text{max}}\). When \(\rho / \rho_{\text{max}}\) is small, the normalised speed \(v / v_{\text{max}}\) can be viewed as linear with normalised density \(\rho / \rho_{\text{max}}\) and the ‘linear factor’, \(m\) captures the linear dependency of speed with density mostly in uncongested traffic flow. When the density is over a critical point, as occurs with congested traffic, the speed drastically reduces with density and tends to zero for maximum density, a condition which may arise at the time of traffic jam when the average traffic speed is very low and traffic density is comparatively high. A power law term can be used in the proposed model to incorporate the effect of congested traffic situation when the normalised density \(\rho / \rho_{\text{max}}\) is high and tends to unity. The term \(n\) used in the proposed model (7) can be regarded as the ‘power factor’ of the speed-density relationship of the traffic model.

As shown in Figure 1(a), normalised speed decreases almost linearly with normalised density up to a critical point where the product of speed and density (which was defined as traffic flow earlier) is a maximum and while the normalised density is above this critical point, the normalised speed reduces very rapidly. The traffic flow versus normalised speed is shown in Figure 1(b) which shows that there is a critical point (critical density) for which the flow is maximum.

Figure 1  Represents the normalised speed vs. normalised density and the flow vs. normalised density using the proposed power law model using \(m = 0.6\) and \(n = 5\) (see online version for colours)

Note: The marked point represents the critical point where flow is at maximum.

4 Relation of proposed model with other basic speed-density model

In this section, we show how the proposed model is generic and how the other basic speed-density functional form of traffic flow (as discussed earlier) can be generated from the proposed model. For the simplification, we have considered the most basic models of speed-density curve, i.e., linear, logarithmic, and exponential models.
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4.1 Linear model

To normalise the linear model (3), we can use the fact that $\rho \to \rho_{\max}$ implies $v \to 0$, hence $\rho_{\max} = 2\rho_0$. Similarly for $\rho \to 0$, $v \to v_{\max}$, hence $v_{\max} = 2v_0$. Now (3) can be normalised as

$$\frac{v}{v_{\max}} = 1 - \frac{\rho}{\rho_{\max}}$$

(8)

Equation (8) is same as the LWR model (1). Now if we put $m = 0$ in the (7) we get (8). Hence, both equations (1) and (3) can be generated from the proposed model (7).

4.2 Logarithmic model

In (5), we can use the similar analogy that $\rho \to \rho_{\max}$ implies $v \to 0$, hence $\rho_{\max} = e\rho_0$. Now, equation (4) can be converted into

$$\frac{v}{v_{\max}} = \frac{v_0}{v_{\max}} \left(1 - \ln \frac{\rho}{\rho_{\max}}\right) = \frac{v_0}{v_{\max}} \left(1 - \ln \frac{\rho}{\rho_{\max}} - \ln e\right) = \frac{v_0}{v_{\max}} \ln \frac{\rho_{\max}}{\rho}$$

(9)

For realistic purposes, we can consider a density for which the speed is $v_{\max}$. Hence, $v_0$ in (9) can be represented as $v_{\max} / \ln \left(\frac{\rho_{\max}}{\rho_{\min}}\right)$ and the logarithmic model (4) can be normalised as

$$\frac{v}{v_{\max}} = \frac{1}{\ln \frac{\rho_{\min}}{\rho_{\max}}} \ln \frac{\rho}{\rho_{\min}}$$

(10)

Table 1  The approximate values of $m$ and $n$ for different value of $\frac{\rho_{\min}}{\rho_{\max}}$ used in the logarithmic model

<table>
<thead>
<tr>
<th>Serial no.</th>
<th>$\frac{\rho_{\min}}{\rho_{\max}}$</th>
<th>$m$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>1.69</td>
<td>0.49</td>
</tr>
<tr>
<td>2</td>
<td>0.02</td>
<td>2.40</td>
<td>0.65</td>
</tr>
<tr>
<td>3</td>
<td>0.03</td>
<td>3.92</td>
<td>0.79</td>
</tr>
<tr>
<td>4</td>
<td>0.04</td>
<td>4.81</td>
<td>0.84</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
<td>5.98</td>
<td>0.88</td>
</tr>
<tr>
<td>6</td>
<td>0.06</td>
<td>7.41</td>
<td>0.91</td>
</tr>
<tr>
<td>7</td>
<td>0.07</td>
<td>8.80</td>
<td>0.93</td>
</tr>
<tr>
<td>8</td>
<td>0.08</td>
<td>11.37</td>
<td>0.95</td>
</tr>
<tr>
<td>9</td>
<td>0.09</td>
<td>12.99</td>
<td>0.96</td>
</tr>
<tr>
<td>10</td>
<td>0.10</td>
<td>15.78</td>
<td>0.97</td>
</tr>
</tbody>
</table>
In the same way, if we have a density \( \rho_{\text{min}} \) for which the speed is \( v_{\text{max}} \) on equation (2), the \( v_f \) can be represented as \( \frac{v_{\text{max}}}{\ln \left( \frac{\rho_{\text{max}}}{\rho_{\text{min}}} \right)} \) and equation (2) can also be transformed into (10). Equation (10) can be converted into the proposed generalised model (7) with proper choice of \( m \) and \( n \). These values depend on the value of \( \frac{\rho_{\text{min}}}{\rho_{\text{max}}} \) as shown in Table 1.

4.3 Exponential model

It is already known that \( \rho \to 0 \) implies \( v \to v_{\text{max}} \) hence using this condition in (4), we get that \( v_{\text{max}} = e v_0 \). Then the normalised of the exponential model (4) can be written as

\[
\frac{v}{v_{\text{max}}} = \frac{1}{e} \exp \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) = \exp \left( -\frac{\rho}{\rho_{\text{max}}} \frac{\rho_{\text{max}}}{\rho_0} \right)
\]

(11)

Considering at density \( \rho_{\text{max}} \), traffic speed is \( v_{\text{min}} \), hence using (11) we can state that \( \rho_0 \) equals to \( \rho_{\text{max}} / \ln \left( \frac{v_{\text{max}}}{v_{\text{min}}} \right) \) and the exponential model can be normalised as

\[
\frac{v}{v_{\text{min}}} = \exp \left( \rho \rho_{\text{max}} \ln v_{\text{min}} \right)
\]

(12)

Equation (12) can be converted into the proposed generalised model (7) with proper choice of \( m \) and \( n \). These values depend on \( \frac{v_{\text{min}}}{v_{\text{max}}} \) as shown in Table 2.

<table>
<thead>
<tr>
<th>Serial no.</th>
<th>( \frac{v_{\text{min}}}{v_{\text{max}}} )</th>
<th>( m )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>10.01</td>
<td>0.89</td>
</tr>
<tr>
<td>2</td>
<td>0.02</td>
<td>12.28</td>
<td>0.92</td>
</tr>
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<td>0.03</td>
<td>12.70</td>
<td>0.92</td>
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<td>4</td>
<td>0.04</td>
<td>13.54</td>
<td>0.94</td>
</tr>
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<td>0.05</td>
<td>14.93</td>
<td>0.95</td>
</tr>
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<td>0.06</td>
<td>17.18</td>
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</tr>
<tr>
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<td>0.07</td>
<td>21.11</td>
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<tr>
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<td>19.33</td>
<td>0.97</td>
</tr>
<tr>
<td>9</td>
<td>0.09</td>
<td>17.68</td>
<td>0.97</td>
</tr>
<tr>
<td>10</td>
<td>0.10</td>
<td>16.12</td>
<td>0.97</td>
</tr>
</tbody>
</table>
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The proposed model can also be used for other speed-density functional forms which are derived or extended version of the linear, logarithmic, and/or exponential model(s). The values of \( m \) and \( n \) can be generated using a least square minimisation method between the proposed functional form and any other functional forms.

5 Extracting parameter values

In the proposed model (7), there are four unknown variables, \( m \), \( n \), \( v_{\text{max}} \), and \( \rho_{\text{max}} \), required to get an approximate relation of traffic speed and density. From the real-time traffic data, we can approximate these values by measuring the speed at four different points of the density. By considering at the instant where the speed is maximum, we can get \( v_{\text{max}} \). Similarly, considering the instant where speed is minimum (approximately equal to 0), we can get \( \rho_{\text{max}} \). Now \( m \) and \( n \) must be computed, hence two measurements can be performed to know the speed where \( \rho \approx \alpha \rho_{\text{max}} \) and to know the density where \( v \approx \beta v_{\text{max}} \), where \( 0 < \alpha, \beta < 1 \). Therefore, the value of \( m \) and \( n \) can be generated from the following approximate equation derived from (7) using simple algebraic manipulation

\[
m \approx \frac{\left( \frac{v}{v_{\text{max}}} \right)}{\rho_{\text{max}}} - (1 - \alpha)
\]

\[
n \approx \frac{\log \left( (1 - \beta) - (1 - m) \left( \frac{\rho}{\rho_{\text{max}}} \right) \right)}{\log \left( \frac{\rho}{\rho_{\text{max}}} \right)_{v = \beta v_{\text{max}}}} - m^{-1}
\]

Due to the existence of a simple functional relationship, the parameters of the proposed model can be derived by using four approximate measurements. This procedure requires the information of \( v_{\text{max}} \) and \( \rho_{\text{max}} \), which sometimes in real-time traffic measurement are not available.

In real-time data, sometimes \( \rho_{\text{max}} \) cannot be obtained directly, but the average maximum speed \( v_{\text{max}} \) can be approximated, since in real-time traffic the traffic speed almost remains constant when traffic density is less and near to zero. Let \( \rho_1, \rho_2, \rho_3 \) be three measured traffic density points where \( 0 < \rho_1 < \rho_2 < \rho_3 < \rho_{\text{max}} \) and the speed or flow data are available for these traffic densities. Considering \( (\rho_1 / \rho_{\text{max}})^n \) can be neglected, we can state that

\[
(\frac{v}{v_{\text{max}}})_{\rho = \rho_1} \approx 1 - (1 - m) (\rho_1 / \rho_{\text{max}})
\]

Since \( v_{\text{max}} \) can be approximated directly from the available traffic data, let define a variable \( \alpha \) to simplify the calculation as follows,

\[
\alpha = \frac{1 - m}{\rho_{\text{max}}} = \frac{1 - (\frac{v}{v_{\text{max}}})_{\rho = \rho_1}}{\rho_1}
\]

The value of \( \alpha \) can be derived from the available data. Using (16), we can state that
Using simple algebraic manipulation the ‘power factor’ $n$ can be approximated as
\[ n \approx \frac{\ln\left(\frac{1 - \alpha \rho_1 - \left(v / v_{\max}\right)_{\rho_{\max}, \alpha}}{1 - \alpha \rho_1 - \left(v / v_{\max}\right)_{\rho_{\max}, \alpha}}\right)}{\ln\left(\rho_2 / \rho_3\right)} \] (18)

The value of $m$ and $\rho_{\max}$ can be approximated using following system of equations
\[ m = \frac{(1 - m)}{\alpha} \] (19)
\[ \rho_{\max} = \left[ m \left(1 - \alpha \rho_2 - \left(v / v_{\max}\right)_{\rho_{\max}, \alpha}\right)\right]^{1/\alpha} \rho_2 \] (20)

The solution of the last two equations can be found numerically or by graphical method.

Interestingly if $\alpha = 0$, the value of $m$ and $\rho_{\max}$ can be estimated easily as $m = 1$ and
\[ \rho_{\max} \approx \left[ \frac{1}{\left(1 - \left(v / v_{\max}\right)_{\rho_{\max}, \alpha}\right)}\right]^{1/\alpha} \rho_2 \] (21)

### 6 Results and discussion

To demonstrate how the proposed model can be used, we have taken the real-time speed-flow data from the International Traffic Database (http://trafficdata.info/). From the data, we sampled four points to estimate the various parameters (described in the previous section).

Figure 2 (a) The normalised density-normalised speed and (b) a representation of normalised density vs. traffic flow behaviour (see online version for colours)

Note: The blue curve shows the actual data derived from the real-time traffic data, the approximated model is represented by the red curve.

As shown in Figure 2(a), the proposed model is used to approximate the relationship between traffic density and the traffic speed, using this approximated relationship, the flow versus traffic density relationship is represented in Figure 2(b). The parameter
7 Conclusions

In this paper, we proposed a simple, explicit model that relates the vehicular speed to the vehicular density on a given road segment. Our proposed model is of a general form and the parameters can be estimated with a very sparse set of data. We demonstrate how the parameters are to be estimated and did the same on a sample dataset. The estimated curves do follow the trends shown by the complete sample of data. Our model can thus be used easily within simulations of traffic on road segments. In this manner, we believe we have made a reasonable contribution to the literature and towards the efforts of modelling vehicular traffic.

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References


